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Inversion-based gait generation for humanoid robots

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***Abstract*— In this paper, we address the problem of gait generation for bipedal robots. We cast the determination of a Center of Mass (CoM) reference trajectory for a given Zero Moment Point (ZMP) desired behaviour as a stable inversion problem for non-minimum phase systems and obtain an analytical solution for any given ZMP trajectory. Our method exploits results from our previous research, in which we derived a family of bounded CoM trajectories associated to a given desired ZMP trajectory.**

I. Introduction

In this paper, we address the problem of synthesis of stable walking gaits for bipedal robots. In general, this problem is too complex to be solved in real-time, and therefore we need to rely either on heavy off-line computations that capture the true robot dynamics (e.g., [1]), or on a combination of high-level approximation schemes coupled with robust low-level feedback control algorithms (e.g., [2]–[5]).

In the latter case, a common approach is to approximate the robot as a simple linear dynamical system, such as a linear inverted pendulum (LIP) or as a cart-table model [2]. At the high level, the task is to compute a trajectory for the system's center of mass (CoM), such that desired performance objectives, often expressed in terms of a desired trajectory of the zero moment point (ZMP), are achieved (*walking pattern generation*). An inverse kinematics problem is solved from the computed CoM trajectory to find trajectories for the individual robot joints. To execute these trajectories on the robot, a stabilizing controller (*stabilizer*) is incorporated to maintain posture, balance, etc. High-level goals are translated into low-level commands that are executed under feedback control. Numerous variations have been developed in recent years, e.g., [3], [6]–[10].

Here, we cast the problem of computing CoM trajectories that achieve a desired ZMP as a stable inversion problem, allowing us to obtain stable, analytical solutions for arbitrary desired ZMP trajectories. We extend our previous work [11] which also considered various special cases of planning individual footsteps and disturbance rejection. Using the concept of *stable inversion*, we are able to generalize our approach, yielding a design framework for both CoM trajectory generation and simultaneous CoM-ZMP planning.

In (e.g., [7], [8], [12]), the problem of simultaneous CoM-ZMP planning has been approached by restricting the class of admissible desired ZMP trajectories to be the family of polynomial splines. Trade-offs between desirable CoM properties and the desired ZMP trajectory can then be handled by relaxing constraints on the desired ZMP trajectory, for example, modifying the coefficients of the first segment of a spline so that CoM continuity conditions are satisfied (e.g., [7]). Preview control has been used in [12] allowing also adjustments of the future ZMP desired trajectory.

Our approach offers a general design methodology which generalizes the idea of capture point of [5].. By casting the problem in terms of stable inversion, we are able to determine CoM trajectories that satisfy boundedness constraints [11], while simultaneously selecting parameters for the ZMP trajectory to achieve ZMP or CoM performance criteria.

The remainder of the paper is organized as follows. After a brief review of the LIP, we introduce our problem and its equivalent formulations in Section II, while the general approach is presented in Section III. In Section IV we discuss a number of related approaches, final value problems and more generally boundary value problems, related compensation schemes, and the use of preview control. In Section V, we present a general design methodology, and provide examples.

II. Preliminaries

For illustration purposes we use only the CoM sagittal plane dynamics approximated by the Linear Inverted Pendulum of Fig. 1 and represented by the second order ODE

$$\ddot{x}_{\rm c}(t) - \omega_o^2 x_{\rm c}(t) + \omega_o^2 x_{\rm zmp}(t) = 0 \tag{1}$$

with $\omega_o = \sqrt{g/h_o}$, $x_{\rm c}$ the center of mass (CoM) x-position and h_o its constant height. Here, $x_{\rm zmp}$ represents the position of the ZMP in a finite-sized foot situation. More detailed treatments can be found in [2], [5], [6].

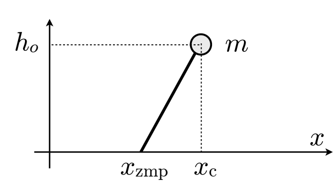

Fig. 1. The Linear Inverted Pendulum (LIP)

For a given assigned desired trajectory $x^{\rm d}_{\rm zmp}(t)$ of the ZMP, the gait generation basic problem consists in finding a CoM trajectory which satisfies the differential equation (1) with $x^{\rm d}_{\rm zmp}(t)$ in place of $x_{\rm zmp}(t)$. If the initial conditions are not fixed, there are infinite CoM trajectories satisfying

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(1) and most of them diverge regardless of the nature of the ZMP. There is therefore an important additional requirement that needs to be stated and that is we want to have the CoM evolution *bounded*. A precise characterisation of the term "bounded" will be given in Section III-C.

In control theory, considering the x_{zmp} as an output function depending upon the CoM state $(x_{\mathrm{c}}, \dot{x}_{\mathrm{c}})$, it is possible to formulate the given problem as an *exact tracking* (satisfying eq. (1) with $x^{\mathrm{d}}_{\mathrm{zmp}}$) with *internal stability* problem (bounded CoM trajectories). A second formulation looks at the inverse system of the original tracking problem i.e. the input becomes an output while the output turns into an input and the system needs to be inverted accordingly. $x^{\mathrm{d}}_{\mathrm{zmp}}$ as a fixed input and we seek for those particular trajectories which remain bounded i.e. for the case at hand, do not diverge exponentially. It is like inverting a proper transfer function but difficulties arise when the original system is non-minimum phase since its zeros become poles of the inverse system. Therefore the inversion needs to be carried out carefully in order to choose the proper bounded trajectories. This is usually defined as a *stable inversion* problem.

III. Exact tracking and stable inversion

We will now illustrate these two problems for the simple model of the humanoid's CoM, similarly to [2].

A. Double integrator/Cart-table

Let us consider a double integrator system $(\mathcal{S}_{\mathcal{I}})$

$$\begin{pmatrix}\dot{x}_{\mathrm{c}}\\ \ddot{x}_{\mathrm{c}}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_{\mathrm{c}}\\ \dot{x}_{\mathrm{c}}\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} u_{\mathrm{c}} \tag{2}$$

and define the output to be

$$y_1 = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_{\mathrm{c}}\\ \dot{x}_{\mathrm{c}}\end{pmatrix} - \frac{1}{\omega_0^2} u_{\mathrm{c}} \tag{3}$$

Note that (2) simply states that $u_{\mathrm{c}} = \ddot{x}_{\mathrm{c}}$ and therefore we recognise the output y_1 to be the ZMP, x_{zmp}, of the Linear Inverted Pendulum. An interesting physical interpretation of this system is the so-called *cart-table model* introduced by Kajita [6]. The corresponding transfer function

$$F_I(s) = \frac{x_{\mathrm{zmp}}}{u_{\mathrm{c}}} = \frac{1 - s^2/\omega_o^2}{s^2} \tag{4}$$

is *non-minimum phase* due to the unstable zero in $s = \omega_o$. It is well-known that, for $(\mathcal{S}_I)$, the control law

$$u_{\mathrm{c}} = \ddot{x}^{\mathrm{d}}_{\mathrm{c}} + k_1(x_{\mathrm{c}} - x^{\mathrm{d}}_{\mathrm{c}}) + k_2(\dot{x}_{\mathrm{c}} - \dot{x}^{\mathrm{d}}_{\mathrm{c}}) \tag{5}$$

with $(x^{\mathrm{d}}_{\mathrm{c}}, \dot{x}^{\mathrm{d}}_{\mathrm{c}})$ suitable state trajectories, solves the output regulation problem with internal stability provided the gains (k_1, k_2) stabilise $(\mathcal{S}_I)$. The suitable $x^{\mathrm{d}}_{\mathrm{c}}$ needs to be a non-exponentially diverging solution of the system $(\mathcal{S}_I)$ driven by $\ddot{x}^{\mathrm{d}}_{\mathrm{c}}$ and such that the output coincides with the desired output $x^{\mathrm{d}}_{\mathrm{zmp}}$ at all times. If the system is initialized in $(x^{\mathrm{d}}_{\mathrm{c}}(0), \dot{x}^{\mathrm{d}}_{\mathrm{c}}(0))$ and driven by $\ddot{x}^{\mathrm{d}}_{\mathrm{c}}$ then the output tracking error will be null for all t thus solving an *exact output tracking* problem. Therefore we define the problem of determining the CoM desired trajectory associated to a given ZMP one, $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$, as an *exact tracking problem*.

The standard linear regulator of Francis [13], where the output reference is generated by an autonomous system, provides an algebraic solution to finding $x^{\mathrm{d}}_{\mathrm{c}}$. For a generic output desired trajectory, $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$ in this context, one approach is to follow the alternative equivalent idea of obtaining a so-called *stable inverse* of our original system $(\mathcal{S}_I)$ driven by the signal $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$. The original idea can be found in a more general setting in [14] (or [15] in a simpler context).

We illustrate this equivalent formulation in the following section while its solution is given in III-C.

B. Linear Inverted Pendulum

The inverse system of $(\mathcal{S}_I)$ can be readily written as $(\mathcal{S}_{II})$

$$\begin{pmatrix}\dot{x}_{\mathrm{c}}\\ \ddot{x}_{\mathrm{c}}\end{pmatrix} = \begin{pmatrix}0 & 1\\ \omega_o^2 & 0\end{pmatrix}\begin{pmatrix}x_{\mathrm{c}}\\ \dot{x}_{\mathrm{c}}\end{pmatrix} + \begin{pmatrix}0\\ -\omega_o^2\end{pmatrix} x_{\mathrm{zmp}} \tag{6}$$

$$y_2 = \begin{pmatrix}\omega_o^2 & 0\end{pmatrix}\begin{pmatrix}x_{\mathrm{c}}\\ \dot{x}_{\mathrm{c}}\end{pmatrix} - \omega_o^2 x_{\mathrm{zmp}} \tag{7}$$

which is recognised to be the Linear Inverted Pendulum having x_{zmp} as *input*. Note that both $(\mathcal{S}_I)$ and $(\mathcal{S}_{II})$ have the same state vector. From the corresponding transfer function

$$F_{II}(s) = \frac{s^2}{1 - s^2/\omega_o^2} = F_I^{-1}(s) \tag{8}$$

it becomes evident how $(\mathcal{S}_I)$ and $(\mathcal{S}_{II})$ are related. The *stable inversion* of $(\mathcal{S}_I)$ problem consists in finding the bounded state trajectories of the system $(\mathcal{S}_{II})$ forced by the input $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$. These correspond to $(x^{\mathrm{d}}_{\mathrm{c}}, \dot{x}^{\mathrm{d}}_{\mathrm{c}})$ and the resulting output turns out to be exactly $\ddot{x}^{\mathrm{d}}_{\mathrm{c}}(t)$. Note that since $(\mathcal{S}_I)$ is non-minimum phase, the inverse system is unstable and thus the generic state trajectory of $(\mathcal{S}_{II})$ is diverging.

For the very simple and particular system here considered, these two equivalent problems represent the two alternative ways of reading the ZMP equation, as an output or an input

$$y_1 = x_{\mathrm{zmp}} = x_{\mathrm{c}} - \frac{1}{\omega_o^2}\ddot{x}_{\mathrm{c}} \tag{9}$$

$$y_2 = \ddot{x}_{\mathrm{c}} = \omega_o^2 x_{\mathrm{c}} - \omega_o^2 x_{\mathrm{zmp}} \tag{10}$$

A simple block diagram of the double integrator with unstable zero-dynamics and ZMP as output is shown in Fig. 2 together with the well-known LIP. In section III-C it will be shown how the stable inversion problem can be solved for any a priori given $x_{\mathrm{zmp}}(t)$ along the lines of [11].

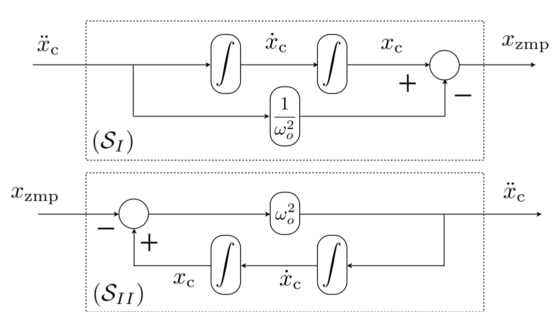

Fig. 2. Direct and Inverse systems: double integrator (cart-table) $\mathcal{S}_I$ and Linear Inverted Pendulum $\mathcal{S}_{II}$

The concepts illustrated so far applied to gait generation are not totally new. Kajita in his seminal paper [6] on preview control clearly states "a walking pattern generation is the inverse problem of this" (Sec 3.1) or "To control the ZMP, it should be the outputs of the system while it appears as the inputs" which is what equations (9) and (10) illustrate. The connection between the two systems $(\mathcal{S}_I)$ (cart-table) and $(\mathcal{S}_{II})$ (LIP) is also illustrated in [2].

Our contribution, so far, is to cast this problem as a stable inverse one and obtain an analytical solution for any given desired ZMP trajectory.

C. Stable Inversion & Boundedness constraint

In this framework, the result of [11] is a solution of the stable inversion problem: find a bounded CoM trajectory associated to a given desired ZMP time evolution $x^{\mathrm{d}}_{\mathrm{zmp}}$ or equivalently find the specific non-diverging solution of the inverse system $(\mathcal{S}_{II})$ – i.e. the LIP – driven by $x^{\mathrm{d}}_{\mathrm{zmp}}$.

Note also that since one pole of $(\mathcal{S}_{II})$ is negative, there are infinite solutions to this problem which differ by an exponentially convergent term. This appears even more evident with the change of coordinates firstly introduced in [9]

$$\begin{pmatrix} x_{\mathrm{u}} \\ x_{\mathrm{s}} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\omega_o} \\ 1 & -\frac{1}{\omega_o} \end{pmatrix} \begin{pmatrix} x_{\mathrm{c}} \\ \dot{x}_{\mathrm{c}} \end{pmatrix}, \quad \begin{pmatrix} x_{\mathrm{c}} \\ \dot{x}_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{\omega_o}{2} & -\frac{\omega_o}{2} \end{pmatrix} \begin{pmatrix} x_{\mathrm{u}} \\ x_{\mathrm{s}} \end{pmatrix} \tag{11}$$

(unstable (diverging) x_{u} and stable (convergent) x_{s} components) leading to the unstable/stable parallel of Fig. 3. Therefore the problem of finding a bounded solution to the inverse system involves necessarily the unstable subsystem

$$\dot{x}_{\mathrm{u}} = \omega_o x_{\mathrm{u}} - \omega_o x^{\mathrm{d}}_{\mathrm{zmp}} \tag{12}$$

while any solution $x_{\mathrm{s}}(t)$ of the stable subsystem

$$\dot{x}_{\mathrm{s}} = -\omega_o x_{\mathrm{s}} + \omega_o x^{\mathrm{d}}_{\mathrm{zmp}} \tag{13}$$

is acceptable. Note that w.r.t. Fig. 2, the output in the parallel of Fig. 3 is just the CoM position. This corresponds to the transfer function

$$F'_{II}(s) = \frac{1}{1 - s^2/\omega_o^2} \tag{14}$$

which retains all the difficulties of $(\mathcal{S}_{II})$.

We can finally state the *boundedness requirement*: find the unique non-exponentially diverging[1] trajectory $x^{\star}_{\mathrm{u}}(t)$ of (12) driven by a given known $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$. As shown in [15] and more in general in [14], the particular solution

$$x^{\star}_{\mathrm{u}}(t; x^{\mathrm{d}}_{\mathrm{zmp}}) = \omega_o \int_0^{\infty} e^{-\omega_o \tau} x^{\mathrm{d}}_{\mathrm{zmp}}(\tau + t) d\tau \tag{15}$$

[1]This is what we define as "bounded". Divergence is acceptable but only if of the same type as the input, as a steady-state behaviour.

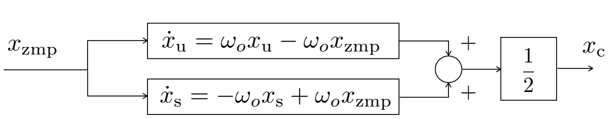

Fig. 3. LIP: Stable & Unstable subsystems

solves the problem. This corresponds to the initial condition

$$x_{\mathrm{u}}(0) = x^{\star}_{\mathrm{u}}(0; x^{\mathrm{d}}_{\mathrm{zmp}}) = \omega_o \int_0^{\infty} e^{-\omega_o \tau} x^{\mathrm{d}}_{\mathrm{zmp}}(\tau) d\tau \tag{16}$$

which will be defined as the *boundedness constraint*.

Some important facts need to be pointed out.

- From (15), we need to know the future values of $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$, i.e. the solution is anticipative. The CoM may need to start moving before the ZMP varies. Standard control design usually does not allow this acausal behaviour unless some special technique is chosen as the preview control used in [6]. Some situations may take advantage of this property as shown in [11].
- Given an analytical expression of $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$ the initial condition and corresponding solution $x^{\star}_{\mathrm{u}}(t)$ can be computed once for all in closed-form. This, together with the chosen trajectory of $x_{\mathrm{s}}(t)$ will define through (11) a state reference trajectory $(x^{\mathrm{d}}_{\mathrm{c}}, \dot{x}^{\mathrm{d}}_{\mathrm{c}})$ and a feedforward $\ddot{x}^{\mathrm{d}}_{\mathrm{c}}$ thus allowing to implement the tracking controller (5) for system $(\mathcal{S}_I)$ or, equivalently, for $(\mathcal{S}_{II})$

$$x_{\mathrm{zmp}} = x^{\mathrm{d}}_{\mathrm{zmp}} + k_3(x_{\mathrm{c}} - x^{\mathrm{d}}_{\mathrm{c}}) + k_4(\dot{x}_{\mathrm{c}} - \dot{x}^{\mathrm{d}}_{\mathrm{c}}) \tag{17}$$

- The boundedness constraint can also be used in a constructive way (see sec. V) by noting that either we have a given fixed $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$ and seek the corresponding $x^{\mathrm{d}}_{\mathrm{c}}(t)$ through the initial condition $x^{\star}_{\mathrm{u}}(0)$ or, alternatively, the ZMP is to be determined so that (16) is satisfied for the current initial conditions i.e. $x_{\mathrm{u}}(0) = x^{\star}_{\mathrm{u}}(0; x^{\mathrm{d}}_{\mathrm{zmp}})$ (which depend from the actual initial position and velocity of the CoM as easily seen from (16)).

Remind that, in this framework, (16) constrains one of the two d.o.f. deriving from the boundary conditions of the second order differential equation (1) so only one d.o.f. is left. This is illustrated in Fig. 4. A useful choice of $x_{\mathrm{s}}(t)$, and therefore of $x_{\mathrm{s}}(0)$ since the input $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$ is fixed, is

$$x_{\mathrm{s}}(0) = 2x_{\mathrm{c}}(0) - x^{\star}_{\mathrm{u}}(0; x^{\mathrm{d}}_{\mathrm{zmp}}) \tag{18}$$

where $x_{\mathrm{c}}(0)$ is the actual initial CoM position. Using relations (11) and being $x_{\mathrm{u}}(0)$ chosen so to satisfy (16), this particular $x_{\mathrm{s}}(0)$ clearly corresponds to letting the desired CoM trajectory start from its actual value $x_{\mathrm{c}}(0)$.

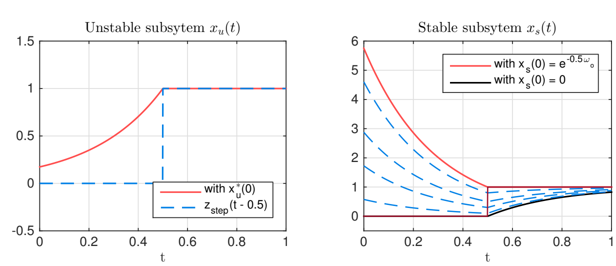

Fig. 4. Bounded trajectories for a step input x_{zmp} starting in $t = 0.5$s: the unique solution $x^{\star}_{\mathrm{u}}(t)$ for the unstable system vs the infinite solutions $x_{\mathrm{s}}(t)$ for the stable system.

The constraint (16), from our point of view, is a fundamental requirement in defining families of state CoM reference trajectories. These are then used in a feedforward plus state error feedback (in its simplest form) control law (5) in the same spirit of [3]. We do not require the real system to start in some specific initial position and velocity CoM values.

Note that the derivation of similar CoM bounded trajectories has been carried out in some specific situations, typically looking for periodic solutions as in [16] or [17].

IV. Discussions

We would first like to explain the difference of the proposed stable inversion approach w.r.t. to some consolidated approaches that we will refer as final value problems. We finally discuss how our approach is strongly connected to the well-known preview design.

A. Final value problems

A number of analytical methods (see [7], [8], [10]) reflect the same fundamental approach firstly illustrated in [7] and based upon the following features.

- Gait planning is focused on finite-time intervals (one or several steps) so the solution is of interest only in $[0,T]$.
- To prevent the solution of an unstable second order system from diverging, instead of formulating an Initial Value Problem where all the necessary boundary conditions to select the unique solution are set at $t=0$, it is extremely useful to set a *final finite condition* in T thus preventing the solution to diverge in finite time. For this reason we define these approaches as *final value*.

In particular, choosing polynomials ZMP trajectories as in e.g., [7], the general solution is taken as the homogeneous solution plus a particular solution which, for the given input, is a polynomial of the same order. Once this general solution is found, the basic approach consists in choosing the remaining unknown coefficients which depend upon the boundary conditions $x_{\mathrm{c}}(0)$ and $x_{\mathrm{c}}(T)$. There are several fundamental remarks w.r.t. to this approach.

- It is true that the solution does not exponentially diverge in the selected interval $[0,T]$ (we refer to a single interval for brevity) but its analytical expression *diverges* after T. Truncating time at T makes the diverging solution viable but we prefer a solution which is guaranteed to be always bounded basically because it respects the main requirement of the humanoid that is to remain in some sort of balance. Forcing the state through the feedforward/feedback interconnection to follow a bounded trajectory appears to be more robust and safe than tracking an exponentially diverging one.
- The resulting bounded CoM trajectory is a *natural* behaviour associated to the given ZMP.
- Assuming to know the CoM desired position in T in order to use it as a final condition $x_{\mathrm{c}}(T)$ is not evident unless some special situation occurs. For example, if the humanoid is required to stop in T, one would choose $x_{\mathrm{c}}(T)$ equal to $x^{\mathrm{d}}_{\mathrm{zmp}}(T)$ but in general the knowledge of the final CoM position may not be so straightforward.

We clarify the fundamental difference between the proposed bounded solution, which will now be defined as x^B_{c}, and the usual analytical solution, defined as x^A_{c}. Since we can make both solutions start from the actual CoM position, $x_{\mathrm{c}}(0)$, the difference is at the terminal value in T. In particular, since $x_{\mathrm{c}}(0)$ is fixed, both solutions coincide if and only if the final CoM position condition in T lies on the bounded solution i.e. $x_{\mathrm{c}}(T)=x^B_{\mathrm{c}}(T)$. Being, for a fixed $x_{\mathrm{c}}(0)$, $x^B_{\mathrm{c}}(T)$ the only bounded solution, if $x_{\mathrm{c}}(T)\neq x^B_{\mathrm{c}}(T)$ the analytical solution $x^A_{\mathrm{c}}(t)$ will grow unbounded. As an example in Fig. 5 the two solutions are reported for a simple ramp input where $x^B_{\mathrm{c}}(T)=2.018\neq x^A_{\mathrm{c}}(T)$ since the boundary condition in T was chosen to be $x^A_{\mathrm{c}}(T)=2$. The slight difference in T makes the analytical solution grow unbounded as t increases.

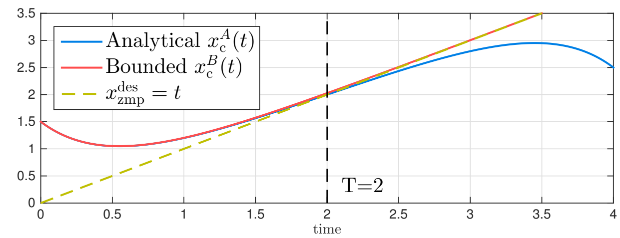

Fig. 5. LIP: analytical and bounded solutions for $T=2$

In [9] it has been noted for the first time that "only the divergent component of motion x_{u} needs to be considered as a boundary condition when generating a gait". Their idea is to modify the ZMP trajectory, which is constrained to be piecewise linear, so that at T the divergent component of motion is at a pre-specified value $x_{\mathrm{u}}(T)$ (for example in order to generate a cyclic gait). A very closely related approach has been carried out in [18].

Summarizing, there are two choices: either one chooses the $x_{\mathrm{u}}(0)$ such that $x_{\mathrm{u}}(T)=x_T$ (but the resulting trajectory will be unbounded) or $x_{\mathrm{u}}(0)=x^\star_{\mathrm{u}}(0;x^{\mathrm{d}}_{\mathrm{zmp}})$ so that the resulting time evolution which is just a state reference, for a given $x^{\mathrm{d}}_{\mathrm{zmp}}(t)$, remains bounded for all t.

B. Boundary Value Problems relaxations

The previous discussion points out, in its simplest form, the existence of two basic Boundary Value Problems:

- use the actual initial position $x_{\mathrm{c}}(0)$ (or alternatively velocity $\dot{x}_{\mathrm{c}}(0)$) and the final position $x_{\mathrm{c}}(T)$ which can also be back-tracked into an equivalent initial velocity (or alternatively position);
- use the actual initial position $x_{\mathrm{c}}(0)$ (or alternatively velocity $\dot{x}_{\mathrm{c}}(0)$) and the boundedness constraint (16) which induces, through (11), an equivalent special initial velocity $\dot{x}^\star_{\mathrm{c}}(0)$ (or alternatively position $x^\star_{\mathrm{c}}(0)$).

However it is possible, as shown in [19], to add flexibility by considering a basis monotone ZMP function with some free parameters and allowing some level of error in the final position and velocity values of the CoM. A quadratic programming problem is then formulated with more constraints. In the same spirit, in [20] relaxation of the boundary conditions are studied in order to compensate disturbances in the CoM state while [10] considers extra intermediate points to enlarge the number of boundary conditions considered.

C. Compensation schemes

As firstly introduced by Park in [21] and extended by [7], [10], [9], [22] or [23], it is possible to compensate the effect of the known swinging leg predefined movement on the total ZMP thus allowing a better correspondence of the simplified LIP model with the true dynamics. The resulting considered system is usually known as 3-mass since it separates the body from the two feet in the definition of the overall ZMP. All the schemes can be represented by the presence of an extra known term $x_{\mathrm{zmp}}^{\mathrm{comp}}(t)$ in the LIP equation

$$\ddot{x}_{\mathrm{c}} - \omega_o^2 x_{\mathrm{c}} = -\omega_o^2 x_{\mathrm{zmp}}^{\mathrm{d}} - \omega_o^2 x_{\mathrm{zmp}}^{\mathrm{comp}} \qquad (19)$$

The term $x_{\mathrm{zmp}}^{\mathrm{comp}}(t)$ is totally known, it contains for example the vertical acceleration or the sagittal displacement of the swinging foot. It is therefore evident that since $x_{\mathrm{zmp}}^{\mathrm{comp}}(t)$ is known, the boundedness constraint can simply be extended with this additional term thus producing from (15) the corresponding contribution in the desired CoM. In all the cited references different approximations have been done, either considering piecewise-linear or periodic functions.

As an illustrative example we report in Fig. 6 the exact equivalent CoM evolution corresponding to a polynomial time trajectory of the swinging mass in the case of neglected vertical foot acceleration. The known swing foot motion in the sagittal plane is shown in Fig. 7.

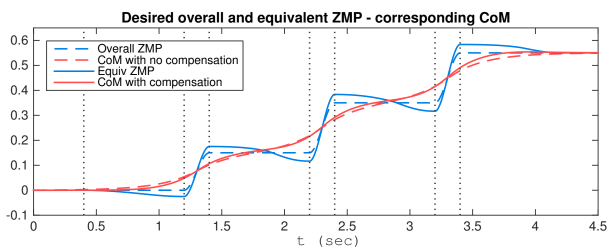

Fig. 6. Compensation: ZMP and corresponding CoM trajectories

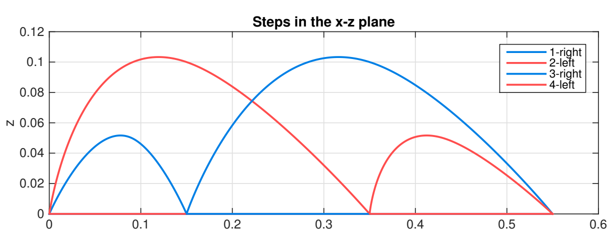

Fig. 7. Swing foot motion in the $x - z$ sagittal plane

D. Preview control

From our point of view the work on preview control applied to humanoid gait planning [6] is the closest w.r.t. the approach here presented. Obviously both consider a tracking problem on the same cart-table system although in [6] an extra integrator is considered making the jerk being the input rather than the acceleration. Moreover the obtained discrete time controller solves a preview control problem with integral action. On the other side the problem we are addressing is equivalent to having an infinite future preview period. Both solutions share the important aspect of having integrated anticipation (non-causality) in their approaches. The presented result however gives a closed-form analytical solution of the CoM trajectory allowing possible real-time implementations. The formal connection of stable inversion of non-minimum phase systems and preview control has been shown [24]. More in particular the resulting CoM trajectory from a preview control approach with control weight tending to zero and infinite preview time coincides with our bounded trajectory with $x_{\mathrm{s}}(0) = 0$ and therefore it is the continuous time version of the interesting results of [25]. The presented framework therefore establishes a natural framework also for these well-known results.

E. Capture Point

We finally would like to point out the strong connections between the concept of "bounded" CoM trajectories for a given desired ZMP evolution and the idea of capture point [5]. As pointed out in [26], the capture point is "a point about which the system is theoretically open-loop stable." With (11), the boundedness constraint (16) becomes

$$x_{\mathrm{u}}(0) = x_{\mathrm{c}}(0) + \frac{\dot{x}_{\mathrm{c}}(0)}{\omega_o} = x_{\mathrm{u}}^{\star}(0; x_{\mathrm{zmp}}) \qquad (20)$$

and we can recognize the extension of the capture point concept to a generic given $x_{\mathrm{zmp}}(t)$.

V. Design

One of the most important aspects of the proposed approach is that we can compute an analytical bounded CoM trajectory for any given ZMP. This contribution, however, seems to require first the choice of the ZMP trajectory so that the corresponding CoM behaviour can be deduced. As introduced in [11] for the point foot case, the basic constraint (16) or (20) can also be used as a design tool.

A. The design principle

Consider a basic monotone function $u_{\mathrm{b}}(t - t_i)$ defined in the interval $[t_{i-1}, t_i]$ which represents the ZMP desired evolution for single step of normalised unit amplitude. We can represent the generic gait, not necessarily cyclic, as a sequence of N basic steps u_{b} of unknown amplitude α_i

$$x_{\mathrm{zmp}}(t) = \sum_{i=1}^{N-1} \alpha_i \left[u_{\mathrm{b}}(t - t_i) + u_{\mathrm{step}}(t - t_i)\right] + \alpha_N u_{\mathrm{b}}(t - t_N)$$

with $u_{\mathrm{step}}(\cdot)$ being the Heaviside step function. The α_i's are still to be determined, means we have to choose the amplitude of each step, i.e. the step length. Since we restrict the family of all possible CoM solutions to the bounded ones, we use the boundedness constraint (20) which states that the actual value of $x_{\mathrm{u}}(0)$, function of the actual CoM initial position and velocity, needs to coincide with the ideal value which is still parametric in the steps amplitude α_i being

$$x_{\mathrm{u}}^{\star}(0;x_{\mathrm{zmp}}) = \sum_{i=1}^{N-1} \alpha_i \left[x_{\mathrm{u}}^{\star}(0;u_{\mathrm{b}}(t-t_i)) + x_{\mathrm{u}}^{\star}(0;u_{\mathrm{step}}(t-t_i))\right] + \alpha_N x_{\mathrm{u}}^{\star}(0;u_{\mathrm{b}}(t-t_N)) \tag{21}$$

By defining the constant pre-computable quantities

$$\begin{aligned} a_i &= \left[x_{\mathrm{u}}^{\star}(0;u_{\mathrm{b}}(t-t_i)) + x_{\mathrm{u}}^{\star}(0;u_{\mathrm{step}}(t-t_i))\right] \\ a_N &= x_{\mathrm{u}}^{\star}(0;u_{\mathrm{b}}(t-t_N)) \quad i=1,\dots,N-1 \end{aligned} \tag{22}$$

the boundedness constraint (20) becomes a linear equation in the unknowns α_i

$$x_{\mathrm{c}}(0) + \frac{\dot{x}_{\mathrm{c}}(0)}{\omega_o} = \sum a_i \alpha_i \tag{23}$$

This leads to a simultaneous CoM/ZMP design in the same spirit of [7] and [12] but with fixed shaped basis ZMP functions. An interesting alternative is shown in [27]. For illustration purposes a Single/Double/Single support ZMP polynomial pattern $u_{\mathrm{b}}(t)$ is shown in Fig. 8.

Together with (23) a number of other interesting constraints can be envisaged.

B. Some interesting constraints

A key role in the constraint selection is how the unknown coefficients appear: as long as the resulting equation is linear in the α_i the solution is straightforward. Our present goal is to illustrate the basic principle by showing some possible choices. We also need to recall that with a fixed $x_{\mathrm{zmp}}^{\mathrm{d}}$, that is no extra d.o.f., we can choose a bounded CoM trajectory starting from the actual position $x_{\mathrm{c}}(0)$ but not also the actual velocity $\dot{x}_{\mathrm{c}}(0)$ (or vice-versa). Note also that the chosen basis (see Fig. 8) ensures continuity in position and velocity at the intermediate t_j's so no extra constraints need to be forced.

We can for example require that the total distance walked during the N steps is L that is

$$\sum_{i=1}^{N} \alpha_i = L \tag{24}$$

since α_i represents, in this discussion, the single step length. First note that, if no other constraint is added, this final distance is approached asymptotically in time and therefore this constraint does not require the CoM to be in L at t_N.

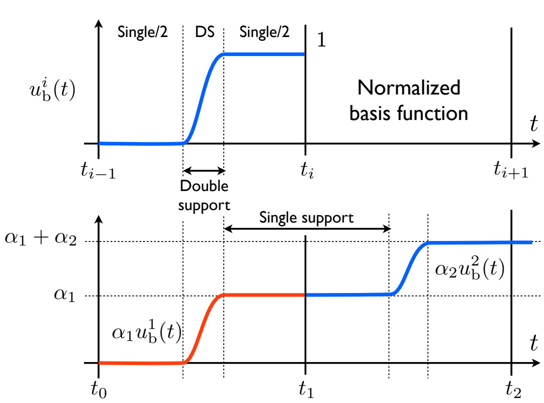

Fig. 8. Design: basis function and its use (2 steps example)

We will illustrate the basic procedure for forcing the CoM position to be in $x_{\mathrm{c}}(t_N)$ at t_N. From the change of coordinates (11) we have

$$\begin{aligned} x_{\mathrm{c}}(t) &= \frac{1}{2}(x_{\mathrm{u}}^{\star}(t;x_{\mathrm{zmp}}) + x_{\mathrm{s}}(t)) \\ &= \frac{1}{2}(x_{\mathrm{u}}^{\star}(t;x_{\mathrm{zmp}}) + e^{-\omega_o t}x_{\mathrm{s}}(0) + \tilde{x}_{\mathrm{s}}(t;x_{\mathrm{zmp}})) \end{aligned} \tag{25}$$

where $\tilde{x}_{\mathrm{s}}(t;x_{\mathrm{zmp}})$ denotes the zero-state response of the stable subsystem. Since we have, from (11),

$$x_{\mathrm{s}}(0) = 2x_{\mathrm{c}}(0) - x_{\mathrm{u}}^{\star}(0;x_{\mathrm{zmp}}) \tag{26}$$

and due to linearity,

$$x_{\mathrm{u}}^{\star}(t;\sum_i \alpha_i u_i) = \sum_i \alpha_i x_{\mathrm{u}}^{\star}(t;u_i) \tag{27}$$

the terms $x_{\mathrm{u}}^{\star}(t;x_{\mathrm{zmp}})$, $x_{\mathrm{u}}^{\star}(0;x_{\mathrm{zmp}})$ and $\tilde{x}_{\mathrm{s}}(t;x_{\mathrm{zmp}})$ will all be linear in the unknown coefficients α_i leading, when setting $x_{\mathrm{c}}(t_N) = x_T$, to another linear equation in the α_i.

Following the same principle, we can easily comply with given initial and/or final velocities $\dot{x}_{\mathrm{c}}(0)$, $\dot{x}_{\mathrm{c}}(t_N)$ or more complicated intermediate constraints like requiring the CoM to be at specific positions at intermediate instants t_j.

C. Discussion

As an example consider the basic unit step depicted in Fig. 8 as a Constant/Spline/Constant ZMP trajectory [7]. The constant section represents the single support phase while the spline function reproduces the translation of the ZMP from one foot to the other during the double support phase. In order to specify a design problem we need to define the constraints on top of the boundedness one which we assume to be the minimal requirement.

In this specific example we have 3 basis functions and therefore we have a total of 2 boundary conditions for the second order differential equation plus 3 coefficients of the basis functions, i.e. the step length for our choice, for a total of 5 d.o.f. to be fixed. The boundedness constraint (16) is a necessary constraint in our framework. This leaves 4 other constraints to be defined. We choose to start from the actual CoM position and velocity, have final CoM zero velocity and walk a total given distance (as in (24)).

The resulting gait of the simultaneous ZMP/CoM design is reported in Fig. 9. Its most interesting aspect is the resulting ZMP which becomes negative first in order to destabilise the system which was starting at rest. Recall that the basis function was chosen to be constant during the single support phase, so the swinging foot could not be used indirectly to make the LIP lean forward. This can only be achieved by making a first backward small step (negative ZMP), let the LIP "fall" forward, make a forward step to decelerate the CoM and finally a last small backward step to bring the LIP at rest with zero velocity at the desired location.

This example shows the great limitation in choosing a constant single support pattern in a basic gait planning scheme since this behaviour does not represent what a human would do having more d.o.f.'s. Note that this is the best policy the LIP can do by using gravity with the given basis ZMP pattern for the given point to point required movement. Intuitively the obtained velocity profile has the lowest possible peak compatible with the duration of the chosen single/double support phases and the chosen basis function. To enrich the possible CoM/ZMP trajectories for the same required movement we could either change the ZMP basis function, allowing a non-constant pattern during the single support as a heel-toe displacement of the ZMP under the supporting foot, or using a swinging foot compensation scheme (or both).

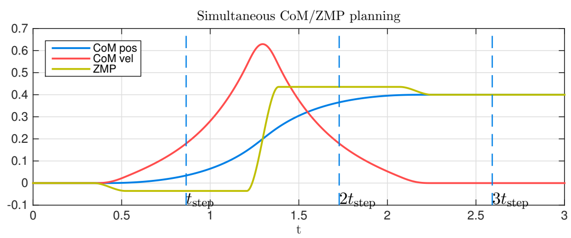

Fig. 9. Design: simultaneous CoM/ZMP example

This simple example shows the potential of the proposed design framework. Possible lines of future research could involve solving some optimisation problem in the choice of the coefficients when fewer constraints are given w.r.t. the number of unknowns (i.e. allowing more steps), characterise other interesting basis functions, enrich the choice of the constraints or even admit some tolerance in their fulfillment.

VI. Conclusions

In this paper, we addressed the problem of synthesis of stable walking gaits for bipedal robots. In particular, we formulated the problem of determining a CoM trajectory for a desired ZMP trajectory as a stable inversion problem. We developed a corresponding design methodology, and gave several example applications.

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